Differentiation

1. Find the tangent to the curve $y = ax^2 + bx + c$ at (x_0, y_0) .

2. Differentiate:

(a) $\dfrac{\sin x + 1}{\sin x - 1}$ **(b)** $\dfrac{\sqrt{1-x^2}}{\sqrt{1+x^2}}$ **(c)** $\ln 2x$

(d) $x \ln x$ **(e)** $x^2 \ln x$ **(f)** x^x

(g) $\ln\left(\dfrac{e^x - e^{-x}}{2}\right)$ **(h)** $\left(\dfrac{x}{a}\right)e^{ax}$ **(i)** $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(j) $\sqrt{x}e^{-\frac{a}{x}}$ **(k)** $\log_k \dfrac{k^x - k^{-x}}{2}$ **(l)** $\ln \ln (a + bx^n)$

(m) $\ln \dfrac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}$ **(n)** $\ln \ln \ln x$ **(o)** $\ln\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)$

(p) $e^{(x^x)}$ **(q)** $e^{\sin x}$ **(r)** $e^x (1 - x^x)$

(s) $\sqrt[a]{x} + \sqrt[x]{a} + \sqrt[x]{x}$ **(t)** $-\dfrac{\cos x}{2\sin^2 x} + \ln\sqrt{\dfrac{1+\cos x}{\sin x}}$

3. Differentiate with respect to x:

(a) $\dfrac{1}{(1-ax)(1-bx)}$ **(b)** $\sqrt{\dfrac{(x-1)(x-2)}{(x+1)^2}}$

(c) $\dfrac{1}{(a+x)^m (b+x)^n}$ **(d)** $\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)^n$

4. Prove that $\sum_{k=2}^{n} k(k-1)C_k^n = n(n-1)2^{n-2}$

5. If $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$, show that $y'' - (\lambda_1 + \lambda_2) y' + \lambda_1\lambda_2 y = 0$

6. If $y = x^n[C_1 \cos (\ln x) + C_2 \sin (\ln x)]$, show that

$$x^2 y'' + (1 - 2n)xy' + (1 + n^2) y = 0$$

7. Find $\dfrac{dy}{dx}$ if $y = 1 - \ln (x + y) + e^y$.

8. Find $\frac{d^n y}{dx^n}$ if $y =$

(a) e^x

(b) $ae^{bx + c}$

(c) a^x

(d) $\ln x$

(e) $\frac{e^x}{x}$

(f) $2^x - \ln x$

(g) $e^{ax} P_n(x)$, where $P_n(x)$ is a polynomial of degree n.

(h) $\frac{\ln x}{x}$

(i) $\frac{1}{\sqrt{ax+b}}$

(j) $x^4 \ln x$

9. Find $\frac{d^n y}{dx^n}$ if $y = a_0 + a_1x + \ldots. + a_nx^n + e^x$.

10. If $x - y + e^{x + y} = 2$, find $\frac{dy}{dx}$.

11. Differentiate with respect to x:

(i) $\sin^{-1} \frac{x^2}{(x^4 + a^4)^{1/2}}$

(ii) $\ln\left[e^x\left(\frac{x-2}{x+2}\right)^{1/2}\right]$

12. If $x = \sin t, y = \sin pt$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$

13. Differentiate:

(a) $\ln\cos\left(\frac{1}{x}\right)$

(b) $\ln\cos\left(\frac{\pi}{4} - x^2\right)$

(c) $\sin^{-1}\frac{x}{x+1}$

(d) $e^{\sin 2x}$

(e) $\sin(\cos x)$

(f) e^{3x^2}

(g) $x^{\cos x}$

(h) a^x , $a > 0$

(i) $\ln(\sin^{-1} x^2)$

(j) $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$

(k) $x\sqrt{a^2 - x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)$

(l) $\cos^{-1}\frac{x^{2n} - 1}{x^{2n} + 1}$

(m) $\sin^{-1}\sqrt{\sin x}$

(o) $\sin^n x \cos^n x$

14. Using differentiation, find the tangents and normals at a point (x_0, y_0) on the of the following curves :

(a) $x^2 + y^2 + 2gx + 2fy + c = 0$

(b) $y^2 = 4ax$

(c) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

(d) $\frac{x}{a} - \frac{y}{b} = 1$

(e) $xy = c^2$

(f) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

15. Show by induction that

$$\sum_{i=1}^{n} \sin ix = \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}} \quad \text{and deduce that} \quad \sum_{k=1}^{n} k\cos kx = \frac{\frac{n+1}{2}\sin\frac{x}{2}\sin\frac{(2n+1)x}{2} - \frac{1}{2}\sin^2\frac{(n+1)x}{2}}{\sin^2\frac{x}{2}}.$$

16. By differentiating $x + x^2 + \ldots + x^n$, find $1^2 + 2^2x + \ldots + n^2 x^{n-1}$.

17. Prove $\cos\frac{x}{2}\cos\frac{x}{4}\ldots\cos\frac{x}{2^n} = \dfrac{\sin x}{2^n \sin\frac{x}{2^n}}$, and hence evaluate: $\frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n}$.

18. If $y = \dfrac{(n+1+x)^{n+1}}{(n+x)^n}$, where n is a fixed positive integer and x is positive,

find $\dfrac{dy}{dx}$ by logarithmic differentiation, and show that y increases with x.

Hence, or otherwise, show that $\left(1+\frac{x}{n}\right)^n < \left(1+\frac{x}{n+1}\right)^{n+1}$.

19. Given that $\dfrac{dy}{dx} = 1\Big/\dfrac{dx}{dy}$. Find $\dfrac{d^2x}{dy^2}$ and $\dfrac{d^2y}{dx^2}$ in terms of $\dfrac{dx}{dy}$.

20. Let $x = f(t)$, $y = g(t)$. Show that $\dfrac{dy}{dx} = \dfrac{dg(t)}{dt}\Big/\dfrac{df(t)}{dt}$.

(a) Hence find the tangent and normal at the point P with parameter t to the curve:

$x = a(2\cos t + \cos 2t)$

$y = a(2\sin t - \sin 2t)$

(b) Show that the tangent at P meets the curve in the points Q, R whose parameters are

$-\frac{1}{2}t$ and $\pi - \frac{1}{2}t$.

(c) Show that $QR = 4a$.

(d) Show that the tangents are Q and R intersect on the circle $x^2 + y^2 = a^2$.

(e) Show that the normals at P, Q, R are concurrent and intersect on the circle: $x^2 + y^2 = 9a^2$.

(f) Show that the equation of the curve is $(x^2 + y^2 + 12ax + 9a^2)^2 = 4a(2x + 3a)^3$.